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Periodic orbits and spectral statistics of pseudointegrable billiards

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We demonstrate for a generic pseudointegrable billiard that the number of periodic orbit families with length less than l increases as $\pi b_0 l^2/\langle a(l)\rangle$, where b_0 is a constant and $\langle a(l)\rangle$ is the average area occupied by these families. We also find that $\langle a(l)\rangle$ increases with l before saturating. Finally, we show that periodic orbits provide a good estimate of spectral correlations in the corresponding quantum spectrum and thus conclude that diffraction effects are not as significant in such studies. [S1063-651X(96)52108-2]

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Billiards are an interesting and well studied class of Hamiltonian systems that display a wide variety of dynamical behavior depending on the shape of the boundary. Of these, pseudointegrable billiards form a subclass and these correspond to rational angled polygonal shaped enclosures with a particle reflecting specularly from the walls [1]. They possess two constants of motion like their integrable counterparts [2] but the invariant surface is topologically equivalent to a sphere with multiple holes [1] and not a torus. As an example, consider the billiard in Fig. 1. For any trajectory, p_x^2 and p_y^2 are conserved. The invariant surface consists of four sheets (copies) corresponding to the four possible momenta $(\pm p_x, \pm p_y)$ that it can have and the edges of these sheets can be identified such that the resulting surface has the topology of a double torus.

The classical dynamics no longer has the simplicity of an integrable system where a transformation to action and angle coordinates enables one to solve the global evolution equations on a torus. On the other hand, the dynamics is nonchaotic with the only interesting feature occurring at the singular vertex with internal angle $3\pi/2$. Here, families of parallel rays split and traverse different paths, a fact that limits the extent of periodic orbit families. This is in contrast to integrable billiards (and to the $\pi/2$ internal angles in Fig. 1) where families of rays do not see the vertex and continue smoothly.

We shall focus here on the periodic orbits of such systems for they form the central object in modern semiclassical theories [3]. Not much is, however, known about the manner in which they are organized and the few mathematical results that exist [4] concern the asymptotic properties of their proliferation rate. For a subclass of rational polygons where the vertices and edges lie on an integrable lattice (the so called almost-integrable systems [4]), these asymptotic results are exact. It is known, for example, that the number of periodic orbit families (those which have an even number of bounces), $N(l)$, increases quadratically with length, l, as $l\to\infty$. For general rational polygons, rigorous results provide bounds on $N(l)$ though it is believed that $N(l)\sim l^2$ even in these cases [4,5]. Numerical verifications of these results are few and perhaps none exist for a general rational polygon that does not belong to the category of almost-integrable systems. Besides, very little is known about other aspects such as the sum rules obeyed by periodic orbits in contrast to the integrable and chaotic limits where these have been well studied [6].

It is not surprising then that periodic orbit theories for polygonal billiards have met with little success, both in quantizing individual levels [7,8] and in explaining the correlations in the quantum spectrum [9]. We shall not deal with the question of determining individual levels here, but merely point out that agreement with the smeared quantum density of states has been observed using complex energies [7,8] although alternate convergent schemes (such as cycle expansions in chaotic systems [10]) are indeed desirable for better resolution of individual levels. In this sense, the full scope of periodic orbit quantization in such systems is largely unknown, even though several researchers have now looked beyond geometric periodic orbit contributions and found evidence of diffractive corrections [11].

Correlations, on the other hand, are related to sum rules obeyed by periodic orbits and these are robust quantities that do not suffer from acute convergence problems. We thus focus on the problem of computing correlations in the quantum spectrum using geometric periodic orbits. Further, our results indicate that these estimates are good, indicating that diffraction effects are not as significant in studies involving the statistical properties of the spectrum [12]. To appreciate this finding however, it is necessary to study the sum rules

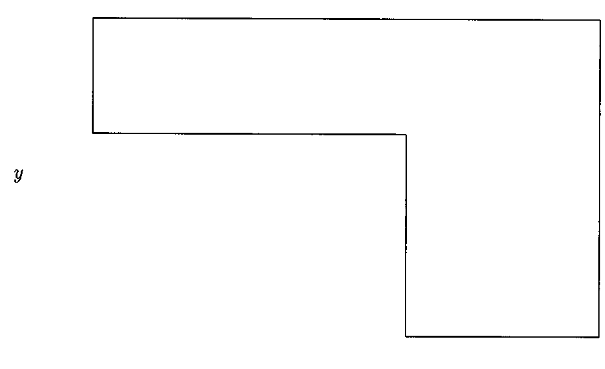

FIG. 1. A pseudointegrable billiard. Periodic orbit families are restricted in extent by the singular ($3\pi/2$) vertex.

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obeyed by periodic orbit families and show that there are important differences from the integrable case contrary to what can be expected from the asymptotic properties of $N(l)$ [8]. To this end, we first verify that periodic orbits in a generic pseudointegrable system obey the sum rule [13] $\langle \Sigma_p \Sigma_{r=1}^{\infty} a_p \delta(l-rl_p)/(rl_p) \rangle = 2\pi b_0$, where b_0 is a constant, $\langle \, \rangle$ denotes the average value, l_p is the length, and a_p the area of a primitive periodic orbit family. This establishes the proliferation law, $N(l)=\pi b_0 l^2/\langle a(l)\rangle$, derived in [13] where $\langle a(l)\rangle$ is the average area occupied by families having lengths $rl_p \leqslant l$. Further, we explore the behavior of $\langle a(l)\rangle$ as a function of length, l, and find that it increases initially before saturating to a value much smaller than the maximum possible area spanned by a single family. The proliferation law, $N(l)$, is thus quadratic only asymptotically and the number of orbits is much larger than that of an equivalent integrable system with the same area. For smaller lengths, $N(l)$ is subquadratic and this is significant for a two-point correlation of the quantum spectrum that we study and for which we demonstrate that periodic orbits provide accurate estimates.

The L-shaped billiard of Fig. 1 that we choose has approximately unit area and has no periodic orbit with an odd number of bounces at the boundary. It does not belong to the class of almost-integrable billiards and is generic in the sense that all sides are irrationally related and periodic orbit lengths are nondegenerate. Unlike some degenerate cases [7] where periodic orbits can be labeled by two integers (analogous to the winding numbers in two-dimensional integrable systems), orbits in generic L-shaped billiards are described by a set of four integers though not every point on the four-dimensional lattice corresponds to a real periodic orbit. This makes it difficult to study sum rules from purely classical considerations. An alternate approach adopted in [13] uses the semiclassical trace formula which expresses the density of quantum energy eigenvalues in terms of periodic orbits [1]:

$$\sum_n \delta(E-E_n) = d_{av}(E) + \frac{1}{4\pi}\sum_p \sum_{r=1}^{\infty} a_p J_0(krl_p) \quad (1)$$

where J_0 is a Bessel function, $\{E_n\}$ and $d_{av}(E)$ are the quantum energy eigenvalues and their average density respectively, and $k=\sqrt{E}$ [14]. For convenience, we have chosen the mass, $m=1/2$ and $\hbar=1$. Starting with

$$g(l)=\sum_n f(\sqrt{E_n};\beta)=\int_{\epsilon}^{\infty} dE \; f(E;\beta)\sum_n \delta(E-E_n) \quad (2)$$

where $f(\sqrt{E};\beta)=J_0(\sqrt{E}l)e^{-\beta E}$ and $0<\epsilon<E_0$, it is possible to show using Eq. (1) that for $\beta \to 0^+$:

$$\sum_p \sum_{r=1}^{\infty} \frac{a_p}{rl_p}\delta(l-rl_p) = 2\pi b_0 + 2\pi \sum_n J_0(\sqrt{E_n}l) \quad (3)$$

where $b_0=\Sigma_p \Sigma_{r=1}^{\infty}(a_p/4\pi)\int_0^{\epsilon} dE \; J_0(\sqrt{E}l)J_0(\sqrt{E}rl_p)$.

It is argued in [13] that b_0 is a constant and we demonstrate this here by plotting $S(l)=\Sigma_p \Sigma_r (a_p/rl_p)$ in Fig. 2 where the summation is restricted to all periodic orbits with $rl_p \leqslant l$. It follows from Eq. (3) that $S(l) \simeq 2\pi b_0 l$.

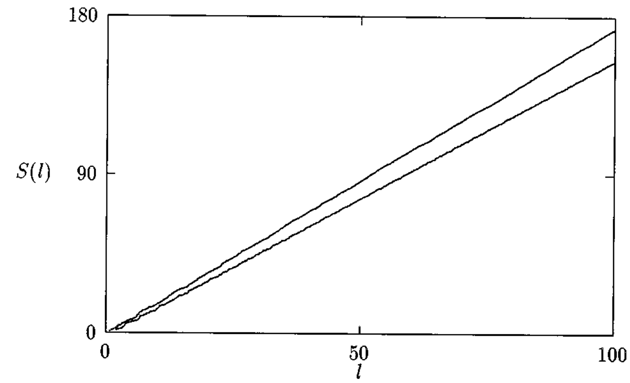

FIG. 2. The function $S(l)=\Sigma_p \Sigma_r (a_p/rl_p)$ for a rectangular and a pseudointegrable (upper curve) billiard. The sum is restricted to orbits with $rl_p \leqslant l$.

Figure 2 shows the behavior of $S(l)$ for a rectangular (integrable) and an L-shaped (pseudointegrable) billiard. In the former case, the orbit lengths l_p can be expressed in terms of the winding numbers (M,N) on the torus [15,16] while the areas $\{a_p\}$ are four times the area, A, of the billiard except for bouncing ball orbits for which they are twice the area. For the pseudointegrable billiard, both the lengths and areas are determined numerically using two different methods. We illustrate one of these by first noting that one member of each periodic orbit family encounters the singular vertex. Further, a nonperiodic orbit originating from the same point but with a momentum slightly different from a periodic orbit suffers a net transverse deviation that equals $(-1)^{n_\phi}\sin(\phi-\phi_p)l_\phi$. Here l_ϕ is the distance traversed by a nonperiodic orbit at an angle ϕ after n_ϕ reflections from the boundary and ϕ_p is the angle at which a periodic orbit exists. These facts can be used to converge on periodic orbits rapidly and the method works for other polygonal billiards. Details of this and the other method employed can be found in [17].

Note that in both cases the curves in Fig. 2 are linear as expected from Eq. (3). For the integrable case, $b_0=0.25$, while for the pseudointegrable billiard $b_0 \simeq 0.27$ [18]. This is the first difference between the two cases. The higher value for the pseudointegrable billiard is possibly due to the fact that at the singular vertex, there can exist more than one periodic orbit with the same value of ϕ_p. It could also reflect diffraction effects that have been neglected in Eq. (1). These issues will be discussed in a future publication [17].

It is clear then that the leading term for the counting function, $N(l)$, is $\pi b_0 l^2/\langle a(l)\rangle$ with corrections provided by the the quantum energy eigenvalues. This result holds for all rational polygons including those which are neither integrable nor almost integrable. Here the average projected phase space area $\langle a(l)\rangle \equiv (\Sigma_p \Sigma_r a_p)/N(l)$ where the summation extends over all orbits with $rl_p \leqslant l$. For rectangular billiards, $\langle a(l)\rangle \simeq 4A$ and this gives the quadratic law for $N(l)$ [3,16]. For pseudointegrable billiards, we plot $\langle a(l)\rangle$ in Fig. 3. The saturation for large l implies $N(l) \sim l^2$ asymptotically [4]. For smaller values of l, $\langle a(l)\rangle$ increases, indicating a (local) subquadratic law for $N(l)$ that we have indeed verified.

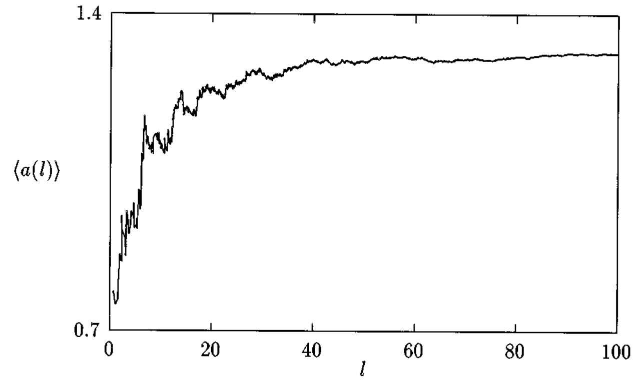

FIG. 3. The average area, $\langle a(l)\rangle$, as a function of orbit length. Note the saturation for long lengths.

Note that the maximum area, a_{max}, occupied by a family is four times the area of the billiard since the invariant surface consists of four sheets corresponding to the four possible momenta a trajectory can have. The value at which $\langle a(l)\rangle$ saturates is thus far smaller than a_{max}. Hence the density of orbit lengths for the pseudointegrable billiard is far in excess of an equivalent integrable billiard having the same area.

With these findings on periodic orbits, we now turn to the statistical analysis of the quantum spectra. A commonly used measure is the spectral rigidity $\Delta(L)$ defined as [19]

$$\Delta(L)=\left\langle \min_{a,b} \frac{d_{av}}{L}\int_{-L/2d_{av}}^{L/2d_{av}} [N(E_0+E)-a-bE]^2 dE \right\rangle \quad (4)$$

where $N(E)$ counts the number of eigenvalues, E_0 is the energy at which the measure is evaluated, and $\langle\ \rangle$ is an averaging in energy over scales larger than the outer scale [15] determined by the slowest frequency of oscillation in Eq. (1). It is possible to analyze the rigidity in terms of periodic orbits using Eq. (1) and the basic semiclassical expression for the rigidity is then [15]

$$\Delta(L)=\left\langle \sum_i \sum_j \frac{A_i A_j}{T_i T_j}\cos(S_i-S_j)H_{ij} \right\rangle \quad (5)$$

where the summations extend over all periodic orbits [20], $A_i=(a_j^2/32\pi^3 l_i\sqrt{E_0})^{1/2}$, $T_i=\partial S_i/\partial E$ evaluated at E_0 and $S_i=\sqrt{E_0}l_i$. The function $H_{ij}=F(y_i-y_j)-F(y_i)F(y_j)-3F'(y_i)F'(y_j)$ where $y_i=LT_i/2d_{av}$ and $F(y)=\sin(y)/y$.

For $L\ll 2\pi d_{av}/T_{min}$ and large E_0, Eq. (5) can be simplified further to yield [15]

$$\Delta(L)=\frac{1}{2\pi^2}\int_0^\infty \frac{d\tau}{\tau^2}K(\tau)G(\pi L\tau) \quad (6)$$

where $G(y)=1-F^2(y)-3F'^2(y)$, $\tau=T/(2\pi d_{av})$, $K(\tau)=2\pi\phi(T)/d_{av}$ and

$$\phi(T)=\left\langle \sum_i \sum_j A_i A_j \cos(S_i-S_j)\,\delta(T-(T_i+T_j)/2) \right\rangle. \quad (7)$$

Equation (6) is useful for analytical studies only when the collective properties of periodic orbits as embodied in $\phi(T)$ are known. A first step in this direction is the diagonal approximation [15] for small T based on the fact that orbit pairs have large action differences and hence off-diagonal terms do not survive averaging. The diagonal sum $\phi_D(T)=\langle \sum_i A_i^2\,\delta(T-T_i)\rangle$ is, however, unknown for generic pseudointegrable systems, though on treating the area, a_i, as constant and using the asymptotic law, $N(l)\sim l^2$, one might infer that $\phi_D(T)$ is constant [8] as in integrable systems [15,6]. These assumptions are, however, incorrect especially for values of T where the diagonal approximation is expected to be valid. We shall therefore investigate this numerically.

For large T, off-diagonal contributions are generally important and more difficult to estimate [21]. For integrable systems where the density of orbit lengths is small, off-diagonal contributions vanish even for large T [15]. One might expect this to be true even in the pseudointegrable situation due to similarities in the asymptotic proliferation laws though it must be noted that the density in the pseudointegrable case is larger compared to an equivalent integrable system having the same area.

In Fig. 4, we plot the function $I(\tau)=\int_0^\tau K(\tau')d\tau'$ (upper curve) as well the diagonal part $I_D(\tau)=\int_0^\tau K_D(\tau')d\tau'$. The fact that $I_D(\tau)$ and $I(\tau)$ coincide until $\tau_c\simeq 0.42$ implies that off-diagonal terms do not contribute for $\tau<\tau_c$. The diagonal approximation thus provides a good estimate of $K(\tau)$ for short times [22].

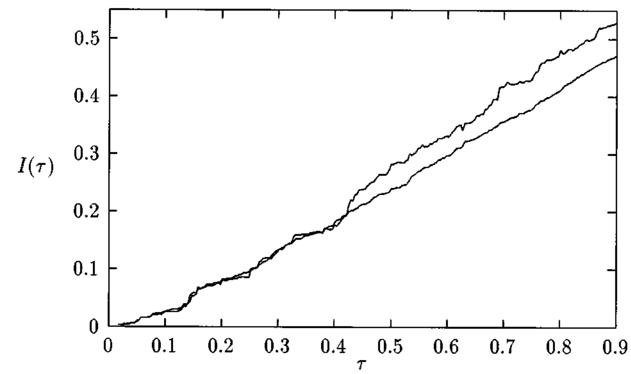

FIG. 4. A plot of $I(\tau)=\int_0^\tau K(\tau')d\tau'$ (upper curve) along with the diagonal part. Note the crossover at $\tau_c\simeq 0.42$ where off-diagonal terms begin to contribute.

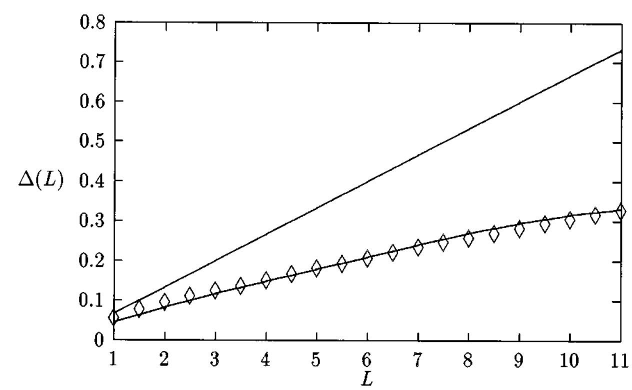

FIG. 5. The spectral rigidity $\Delta(L)$ for the L-shaped billiard. The lower curve is evaluated using periodic orbits while the diamonds are the values computed directly using energy eigenvalues. The straight line above is the Poisson result ($L/15$).

The important departure [as far as $\Delta(L)$ is concerned] from integrable behavior lies in the fact that $I_D(\tau)$ displays variations in slope for $\tau \leq \tau_c$ indicating that $K_D(\tau)$ is not a constant [23]. For $\tau > \tau_c$, $K_D(\tau)$ is constant but has a value much smaller than unity [for integrable systems $K_D(\tau)=1$ for all values of τ]. Of significance as well is the nonvanishing contribution of the off-diagonal part $[I(\tau)-I_D(\tau)]$ for $\tau > \tau_c$. In this example a power law proliferation of periodic orbits gives rise to nonzero off-diagonal contributions in the form factor.

Finally, we compute the spectral rigidity using Eq. (5) with 6621 periodic orbits and estimate the contribution of the longer ones using Eq. (6) and an interpolation for $K(\tau)$. The result is displayed in Fig. 5 where we also plot $\Delta(L)$ obtained numerically using the quantum eigenvalues. The agreement is remarkably good leading to the conclusion that diffraction effects are not as significant for the statistical properties of the spectrum.

In summary, we have brought to light several interesting properties of periodic orbits in pseudointegrable billiards and numerically established that periodic orbits provide good estimates of spectral correlations even when diffraction plays a role. Our specific conclusions are the following.

(i) Orbit families obey the sum rule $\langle \Sigma_p \Sigma_{r=1}^{\infty} a_p \delta(l - r l_p)/(r l_p) \rangle = 2\pi b_0$ thereby giving rise to the proliferation law $N(l) = \pi b_0 l^2 / \langle a(l) \rangle$ for all rational polygons.

(ii) $\langle a(l) \rangle$ increases initially before saturating to a value much smaller than a_{max}; the asymptotic proliferation law is thus quadratic even for systems that are not almost-integrable and the density of periodic orbits lengths is far greater than an equivalent integrable system having the same area.

(iii) The diagonal part of the form factor $K(\tau)$ approaches a constant at $\tau = \tau_c$ and the value is much smaller than unity. In contrast, the diagonal part in integrable systems is identically equal to 1.

(iv) Off-diagonal contributions in $K(\tau)$ are nonzero for $\tau > \tau_c$ even though the proliferation of periodic orbits is quadratic as in integrable billiards.

(v) Periodic orbits provide good estimates of correlations in the quantum spectrum.

It is a pleasure to acknowledge useful discussions with Predrag Cvitanović, Bertrand Georgeot, Gregor Tanner, and Niall Whelan.

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